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# Dividing and Forking : A Proof of the Equivalence (Model theoretic aspects of the notion of independence and dimension)

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# Dividing and Forking

## – A Proof of the Equivalence –

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### 1 Introduction and Preliminaries

Let  $T$  be a simple complete theory. Then the two notions forking and dividing are equivalent. (See [1].) The usual proof of this fact needs Erdős-Rado theorem, a basic result in combinatorial set theory. Erdős-Rado theorem is a theorem on uncountable cardinals, extending Ramsey's theorem. So it is somewhat strange to use such a theorem for proving the fact when the language is countable. In this article, we present a new proof that will only use compactness and a Ramsey-type argument.

We explain the notations in this article and recall some basic definitions.  $L$  is a language and  $T$  is a complete theory formulated in  $L$ . Although the countability of  $L$  is not necessary, we assume that  $L$  is countable for simplicity. We fix a big saturated model  $\mathcal{M}$  of  $T$  and we work in  $\mathcal{M}$ . Small subsets of  $\mathcal{M}$  are denoted by  $A, B, \dots$ . Finite tuples in  $\mathcal{M}$  are denoted by  $a, b, \dots$ . Variables are  $x, y, \dots$ . Formulas are denoted by  $\varphi, \psi, \dots$ . If all the free variables of  $\varphi$  are contained in  $x$ ,  $\varphi$  is sometimes written as  $\varphi(x)$ . For a set  $A$ ,  $L(A)$  is the language  $L$  augmented by the names (constants) for  $a \in A$ . For simplicity of the notation, we write  $\varphi \in L(A)$ , if  $\varphi$  is a formula in  $L(A)$ . In general, a formula  $\varphi \in L(A)$  has the form  $\psi(x, a)$ , where  $\psi(x, y)$  is an  $L$ -formula with  $xy$  free and  $a$  is the list of parameters (from  $A$ ) appearing in  $\varphi$ .  $a$  will be called the parameters of  $\varphi$ .

A sequence  $\{a_i : i \in \omega\}$  is called an indiscernible sequence over  $A$ , if for any strictly increasing  $f : \omega \rightarrow \omega$ , there is an automorphism  $\sigma$  of  $\mathcal{M}$  that extends the mapping  $id_A \cup \{\langle a_i, a_{f(i)} \rangle\}_{i \in \omega}$ . We say that  $\{a_i : i \in \omega\}$  starts with  $a$ , if  $a_0 = a$ .

**Definition 1.** A formula  $\varphi(x, a)$  divides over  $A$  if there is an indiscernible sequence  $\{a_i : i \in \omega\}$  starting with  $a$  such that  $\{\varphi(x, a_i) : i \in \omega\}$  is inconsistent.

A set  $\Phi$  of formulas is said to be  $k$ -inconsistent, if every subset  $\Psi_0 \subset \Phi$  of size  $k$  is inconsistent. If  $\varphi(x, a)$  divides over  $A$ , by the indiscernibility of  $\{a_i : i \in \omega\}$ , there is some  $k \in \omega$  such that  $\{\varphi(x, a_i) : i \in \omega\}$  is  $k$ -inconsistent. In this case we say that  $\varphi(x, a)$   $k$ -divides over  $A$ .

**Definition 2.** A formula  $\varphi(x, a)$  forks over  $A$  if it is covered by a finite number of dividing formulas, more precisely, if there is a finite number of formulas  $\psi_i(x, b_i)$  ( $i = 1, \dots, n$ ) with the following properties:

1.  $\mathcal{M} \models \forall x[\varphi(x, a) \rightarrow \bigvee_{i=1, \dots, n} \psi_i(x, b_i)]$ ;
2. Each  $\psi_i(x, b_i)$  divides over  $A$ .

$T$  is called simple if there is a bound for the length of a dividing sequence of complete types. The simplicity of  $T$  is equivalent to the finiteness of the rank defined below:

**Definition 3.** Let  $\Sigma(x)$  be a set of formulas with parameters (with  $x$  free). Let  $\Phi(x, y)$  be a finite set of  $L$ -formulas and let  $k \in \omega$ . The rank  $D(\Sigma(x), \Phi(x, y), k)$  is defined by:

1.  $D(\Sigma(x), \Phi(x, y), k) \geq 0$  if  $\Sigma(x)$  is consistent;
2.  $D(\Sigma(x), \Phi(x, y), k) \geq \alpha + 1$  if there is  $a$  and  $\varphi \in \Phi$  such that  $D(\Sigma(x) \cup \{\varphi(x, a)\}, \Phi(x, y), k) \geq \alpha$  and such that  $\varphi(x, a)$   $k$ -divides over the parameter set of  $\Sigma$ ;
3.  $D(\Sigma(x), \Phi(x, y), k) \geq \delta$  (a limit ordinal) if  $D(\Sigma(x), \Phi(x, y), k) \geq \alpha$  for any  $\alpha < \delta$ .

In the same manner,

## 2 Simple theories

In what follows,  $T$  is a simple complete theory. Let us begin with the following lemma. A proof here is essentially the same as the one presented in Ziegler's book [3].

**Lemma 4.** *Let  $\varphi(x) \in L(A)$ . Then  $\varphi(x)$  does not fork over  $A$ .*

*Proof.* For simplicity we assume  $A = \emptyset$ . Suppose otherwise and choose  $\psi_i(x, b)$  ( $i = 1, \dots, n$ ) and  $k \in \omega$  such that

1. each  $\psi_i(x, b)$   $k$ -divides over  $\emptyset$ ;
2.  $\forall x(\varphi(x) \rightarrow \bigvee_{i=1, \dots, n} \psi_i(x, b))$  holds.

Then we choose  $n_1, \dots, n_m \leq n$  and  $b_1, \dots, b_m$  (copies of  $b$ ) such that

3.  $\psi_{n_i}(x, b_i)$   $k$ -divides over  $\{b_j : j < i\}$ , for each  $i = 1, \dots, m$ ;
4.  $\varphi(x) \wedge \bigwedge_{i=1, \dots, m} \psi_{n_i}(x, b_i)$  is consistent, and its  $D(*, \{\psi_i : i = 1, \dots, n\}, k)$ -rank is minimum among such.

By moving the  $b_i$ 's, we can assume that each  $\psi_{n_i}(x, b_i)$   $k$ -divides over  $\{b\} \cup \{b_j : j < i\}$ . By conditions 2 and 4, there is  $n_{m+1} \leq n$  such that

$$\varphi(x) \wedge \bigwedge_{i=1, \dots, m} \psi_{n_i}(x, b_i) \wedge \psi_{n_{m+1}}(x, b) \text{ is consistent.}$$

Since  $\psi_{n_{m+1}}(x, b)$  divides, by letting  $b_{m+1} = b$ , we have

$$D(\varphi(x) \wedge \bigwedge_{i=1, \dots, m} \psi_{n_i}(x, b_i), \Psi, k) > D(\varphi(x) \wedge \bigwedge_{i=1, \dots, m+1} \psi_{n_i}(x, b_i), \Psi, k)$$

where  $\Psi = \{\psi_i : i = 1, \dots, n\}$ . This contradicts our choice of  $n_i$  ( $i \leq m$ ) and  $b_i$  ( $i \leq m$ ) (condition 4).  $\square$

*Remark 5.* 1. Let  $A \subset B$  and  $p(x) \in S(A)$ . Then there is an extension  $q(x) \in S(B)$  of  $p(x)$  such that  $q(x)$  does not divide over  $A$ . This can be shown as follows: Let  $\Gamma(x) = p(x) \cup \{\neg\varphi(x) \in L(B) : \varphi(x) \text{ does not divide over } A\}$ . Then  $\Gamma(x)$  is consistent, since otherwise we would have  $p(x) \vdash \varphi_1(x) \vee \cdots \vee \varphi_n(x)$ , for some  $\varphi_i$  dividing over  $A$ . So  $p(x) \in S(A)$  forks over  $A$ , contradicting the above lemma. Choose  $a \models \Gamma$ , and let  $q(x) = \text{tp}(a/B)$ . Then, clearly  $q(x)$  does not divide over  $A$ .

2. Suppose that  $\text{tp}(a/Abc)$  does not divide over  $A$  and that  $\text{tp}(b/Ac)$  does not divide over  $A$ . Then  $\text{tp}(ab/Ac)$  does not divide over  $A$ : Let  $\varphi(x, y, c) \in \text{tp}(ab/Ac)$ . Let  $I = \{c_i : i \in \omega\}$  be an arbitrary indiscernible sequence with  $c_0 = c$ . Since  $\text{tp}(b/Ac)$  does not divide over  $A$ , there is  $b'$  (a copy of  $b$  over  $Ac$ ) such that  $I$  is  $Ab'$ -indiscernible. For an  $A$ -automorphism  $\sigma : b' \mapsto b$ ,  $\sigma(I)$  is an  $Ab$ -indiscernible sequence. Notice then that  $J = \{b\sigma(c_i) : i \in \omega\}$  is an  $A$ -indiscernible sequence with  $b\sigma(c_0) = bc$ . Since  $\text{tp}(a/Abc)$  does not divide over  $A$ , there is  $a'$  (a copy of  $a$  over  $A$ ) such that  $a' \models \bigwedge_{d \in J} \varphi(x, d)$ . So  $\sigma^{-1}(a') \models \bigwedge_{i \in \omega} \varphi(x, b', c_i)$ . In particular,  $\{\varphi(x, y, c_i) : i \in \omega\}$  is satisfiable.

**Lemma 6.** *For each non-algebraic type  $p(x) \in S(A)$ , there is an  $A$ -indiscernible sequence  $J = \{b_i : i \in \omega\}$  in  $p$  such that  $\text{tp}(J \setminus \{b_0\}/Ab_0)$  does not divide over  $A$ .*

*Proof.* First we inductively choose  $a_i$ 's realizing  $p$  such that, for each  $i \in \omega$ ,

$$\text{tp}(a_i/A_i) \text{ does not divide over } A,$$

where  $A_i = A \cup \{a_j\}_{j < i}$ . Then, by an iterative use of Remark above,  $\text{tp}(\{a_j\}_{j > 0}/Aa_0)$  does not divide over  $A$ . Similarly we can show that  $\text{tp}(\{a_j\}_{j > i}/Aa_i)$  does not divide over  $A$ , for each  $i$ .

Now let  $\Gamma(\{x_i : i \in \omega\})$  be the following set of  $L(A)$ -formulas:

$$\bigcup_{i \in \omega} p(x_i) \cup \bigcup_{i \in \omega, F \subset \omega \setminus i} \{\neg\varphi(x_F, x_i) : \varphi(x_F, a_0) \text{ divides over } A\},$$

where  $x_F = x_{i_0}, \dots, x_{i_k}$  if  $F = \{i_0 < \cdots < i_k\}$ . Clearly  $\Gamma$  is realized by  $I = \{a_i : i \in \omega\}$ . Moreover, since each  $a_i$  realizes  $p$ , any infinite subsequence of  $I$  realizes  $\Gamma$ . In other words,  $\Gamma$  has the subsequence property. So there is an  $A$ -indiscernible sequence  $J = \{b_i : i \in \omega\}$  realizing  $\Gamma$ . It is clear that  $\text{tp}(J \setminus \{b_0\}/Ab_0)$  does not divide over  $A$ .  $\square$

**Lemma 7.** *Suppose that  $\varphi(x, a)$  divides over  $A$ . Let  $p(x) = \text{tp}(a/A)$  and choose an  $A$ -indiscernible sequence  $J = \{b_i : i \in \omega\}$  in  $p$  having the property described in Lemma 6. Then  $\{\varphi(x, b_i) : i \in \omega\}$  is inconsistent.*

*Proof.* Choose  $k$  such that  $\varphi(x, a)$   $k$ -divides over  $A$ , and choose an  $A$ -indiscernible sequence  $I = \{a_i : i \in \omega\}$  such that  $\{\varphi(x, a_i) : i \in \omega\}$  is  $k$ -inconsistent. By moving  $J$  by an  $A$ -automorphism, we may assume that  $b_0 = a_0$ . Since  $\text{tp}(J \setminus \{b_0\}/b_0)$  does not divide, there is  $\{b'_i : i > 0\}$  (a copy of  $J \setminus \{b_0\}$  over  $Ab_0$ ) such that

$$a_i\{b'_i : i > 0\} \equiv_A a_j\{b'_i : i > 0\} \equiv_A J$$

holds for any  $i, j$ . Moreover, by Ramsey's theorem, we can assume that  $I$  is indiscernible over  $A\{b'_i : i > 0\}$ .

**Claim A.**  $\Phi(x) = \{\varphi(x, b_i) : i \in \omega\}$  is  $k$ -inconsistent.

Suppose otherwise and let  $\alpha = D(\Phi(x), \varphi(x, y), k)$ . Since  $J$  is an indiscernible sequence, we have  $J \equiv_A J \setminus \{b_0\} \equiv_A \{b'_i : i > 0\}$ . So we have

$$\alpha = D(\{\varphi(x, b_i) : i > 0\}, \varphi(x, y), k).$$

However,  $\varphi(x, a_0)$  divides over  $A\{b'_i : i > 0\}$ , so we must have  $D(\{\varphi(x, a_0)\} \cup \{\varphi(x, b_i) : i > 0\}, \varphi(x, y), k) < \alpha$ . This is a contradiction.  $\square$

**Proposition 8.** For  $i = 1, \dots, m$ , let  $\varphi_i(x, a)$  be a formula that divides over  $A$ . Then  $\bigvee_{i=1, \dots, m} \varphi_i(x, a)$  divides over  $A$ .

*Proof.* Choose  $J$  as in Lemma 7, then for each  $i$  there is  $k_i$  such that  $\{\varphi_i(x, b) : b \in J\}$  is  $k_i$ -inconsistent. Let  $k = \max\{k_1, \dots, k_m\}$ . Then  $\{\bigvee_{i=1, \dots, m} \varphi_i(x, b) : b \in J\}$  is  $mk$ -inconsistent. Hence  $\bigvee_{i=1, \dots, m} \varphi_i(x, a)$  divides over  $A$ .  $\square$

**Lemma 9.** Suppose that  $p(x) \in S(A)$  does not divide (fork) over  $A_0 \subset A$ . For any  $B \supset A$ , there is a  $\models p$  such that  $\text{tp}(a/B)$  does not divide over  $A_0$ .

*Proof.* Let  $\Psi(x)$  be the following set of  $L(B)$ -formulas:

$$p(x) \cup \{\neg\varphi(x) \in L(B) : \varphi(x) \text{ divides over } A_0\}.$$

$\Psi(x)$  is consistent, since otherwise we would have that  $p(x)$  forks over  $A_0$ . Let  $a \models \Psi(x)$ . Then it follows that  $\text{tp}(a/B)$  does not divide over  $A_0$ .  $\square$

**Proposition 10** (Symmetry).  $\text{tp}(a/Ab)$  does not divide over  $A \Rightarrow \text{tp}(b/Aa)$  does not divide over  $A$ .

*Proof.* First we inductively choose  $a_i$ 's such that

- $a_i \models \text{tp}(a/Ab)$ ;
- $\text{tp}(a_i/A \cup \{a_j : j < i\})$  does not divide over  $A$ .

This process can be done by an iterative use of Lemma 9. As in the proof of Lemma 7, by compactness, we can assume that  $I = \{a_i : i \in \omega\}$  is an  $A$ -indiscernible sequence satisfying the conditions

1.  $\text{tp}(\{a_i : i > 0\}/Aa_0)$  does not divide over  $A$  ( $i \in \omega$ );
2. By letting  $q_a(x) = \text{tp}(b/Aa)$ ,  $b$  is a common solution of  $q_{a_i}(x)$  ( $i \in \omega$ ).

Thus  $q_a(x)$  does not divide over  $A$ , by Lemma 7.  $\square$

## References

- [1] Byunghan Kim, Forking in simple unstable theories, *Journal of the London Mathematical Society*, vol. 57, 257-267 (1998).
- [2] Byunghan Kim and Anand Pillay, Simple theories, *Annals of Pure and Applied Logic*, vol 88, 149-164 (1997).
- [3] Martin Ziegler, “A Course in Model Theory”, Cambridge University Press (2012).
- [4] Frank O. Wagner, “Simple Theories”, Springer Netherlands; Softcover reprint of hardcover 1st ed. (2010).